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Numerical Solution of Inverse Radiative–Conductive Transient Heat Transfer Problem in a Grey Participating Medium

J. Zmywaczyk · P. Koniorczyk

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Abstract The problem of simultaneous identification of the thermal conductivity $\Lambda(T)$ and the asymmetry parameter g of the Henyey–Greenstein scattering phase function is under consideration. A one-dimensional configuration in a grey participating medium with respect to silica fibers for which the thermophysical and optical properties are known from the literature is accepted. To find the unknown parameters, it is assumed that the thermal conductivity $\Lambda(T)$ may be represented in a base of functions $\{1, T, T^2, \ldots, T^K\}$ so the inverse problem can be applied to determine a set of coefficients $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_K; g\}$. The solution of the inverse problem is based on minimization of the ordinary squared differences between the measured and model temperatures. The measured temperatures are considered known. Temperature responses measured or theoretically generated at several different distances from the heat source along an x axis of the specimen set are known as a result of the numerical solution of the transient coupled heat transfer in a grey participating medium. An implicit finite volume method (FVM) is used for handling the energy equation, while a finite difference method (FDM) is applied to find the sensitivity coefficients with respect to the unknown set of coefficients. There are free parameters in a model, so these parameters are changed during an iteration process used by the fitting procedure. The Levenberg– Marquardt fitting procedure is iteratively searching for best fit of these parameters. The source term in the governing conservation-of-energy equation taking into account absorption, emission, and scattering of radiation is calculated by means of a discrete ordinate method together with an FDM while the scattering phase function approximated by the Henyey–Greenstein function is expanded in a series of Legendre polynomials with coefficients $\{c_l\} = (2l + 1)g^l$. The numerical procedure

J. Zmywaczyk (✉) · P. Koniorczyk
Section of Aerodynamics and Thermodynamics, Faculty of Mechatronics,
Military University of Technology, 00-908 Warsaw, Poland
e-mail: janusz.zmywaczyk@wat.edu.pl

proposed here also allows consideration of some cases of coupled heat transfer in non-grey participating media. The inverse method may be treated, after performing a suitable validation, as an alternative method in relation to other classical measurement methods for investigation of thermophysical parameters of solid states.

Keywords Insulation materials · Inverse method · Radiative properties · Thermal conductivity

1 Introduction

Identification of thermophysical properties of fibrous insulating materials like silica wool by using an inverse method is a challenging task. First, this is because of the complexity of the mathematical formulation of the problem in which various heat transfer modes should be taken into account to obtain a physically correct temperature distribution inside the medium. Second, the inverse methods belong to the class of ill-posed problems in a Hadamard sense, meaning that the solution may not exist, or may not be unique, or small errors in the initial data can result in much larger errors in the answers. To minimize the drawbacks of the ill-conditioned inverse problems, two main mathematical approaches are commonly used: The Tikhonov regularization method and iterative regularization method [1–3]. With reference to classical methods of determining thermophysical properties of materials, the inverse methods can be used to estimate simultaneously from a single experiment a few parameters, e.g., the thermal conductivity Λ and volumetric heat capacity (ρc_p), which can be also temperature dependent [4,5]. In the literature, there are many papers devoted to the solution of inverse heat conduction problems, but there are considerably fewer papers dealing with identification of thermophysical and/or optical parameters governing transient heat transfer involving radiation and conduction. Liu et al. [6] used discrete ordinate method (DOM) to solve the direct problem (DP) in one-dimensional semitransparent plane-parallel media with opaque and reflecting boundaries and then the conjugate gradient method (CGM) to determine the inhomogeneous source term from the specified incident radiation intensities on the boundaries as a result of an inverse problem solution. Li [7] studied an inverse conduction–radiation problem for simultaneous estimation of the single scattering albedo ω_0, the optical thickness τ_0, the conduction-to-radiation parameter N, and the scattering phase function from the known exit radiation intensities. The DP was solved using the method of spherical harmonics p_N- approximation [8, Chap. 15], while the CGM was used to solve the inverse problem. A genetic algorithm was used by Li and Yang [9] to solve an inverse radiation problem for simultaneous estimation of the single scattering albedo ω_0, the optical thickness τ_0, and the phase function from a knowledge of the exit radiation intensities.

In this paper, an inverse approach, based on the Levenberg–Marquardt minimization method [1], was used for simultaneous identification of the temperature-dependent thermal conductivity $\Lambda(T)$ and the asymmetry parameter g of the Henyey–Greenstein phase function for the case of one-dimensional transient radiative–conductive heat transfer in a semitransparent plane-parallel medium.

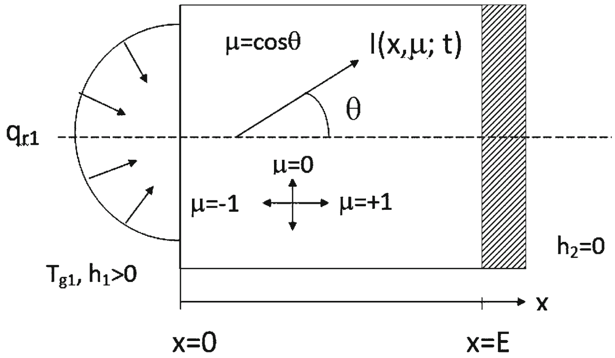

Fig. 1 Physical model of the conductive–radiative heat transfer in a planar slab

2 Problem Formulation

Transient coupled conduction and radiation heat transfer are considered in a fictitious insulation material characterized by thermophysical and optical properties typical for a fibrous insulation made of silica fibers and air with a medium density ρ of $20\,\mathrm{kg \cdot m^{-3}}$ and a specific heat $c_p = 670\,\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$ [10].

2.1 Direct Problem

To find a solution of the direct problem schematically presented in Fig. 1, the following simplifying assumptions were accepted:

- Transient heat transfer by coupled conduction and radiation is considered only in a one-dimensional plane-parallel medium of thickness E
- The medium is homogenous, semitransparent, and grey for wavelengths between $1\,\mu\mathrm{m}$ and $100\,\mu\mathrm{m}$
- The boundaries of the slab are diffusely emitting, absorbing, and reflecting thermal radiation with constant emissivity $\varepsilon_1 = \varepsilon_2 = 0.9$
- Heat conduction within the medium is governed by the temperature-dependent thermal conductivity, $\Lambda(T)$, and by the constant volumetric heat capacity, ρc_p
- Transport of thermal radiation through the medium is due to absorption, emission, and anisotropic scattering
- The radiation intensity $I(x, \mu; t)$ does not depend on the polar angle φ (see [8, p. 426])
- The scattering phase function $P(\mu' \to \mu)$ is given by the Henyey–Greenstein phase function with the asymmetry parameter g
- The refractive index n of the medium is equal to 1 (see [10])
- Initially at time $t = 0$, the slab is in a state of thermodynamical equilibrium with surroundings at a constant temperature T_{g1} which equals $T(x, 0) = T_0 = T_{g1} = 300\,\mathrm{K}$

– Radiation heat flux q_{r1} is generated by the hemispherical surface of the blackbody of constant temperature $T_{b,1} = 400\,\mathrm{K}$ incident upon the face $x = 0$ of the slab within the time interval $0 < t < t_h$, where t_h is the time of heating
– There is free convection of constant heat transfer coefficients $h_1 = 5\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$ at the face $x = 0$ of the slab while its face $x = E$ is adiabatic
– The mean values of the absorption and scattering coefficients are $\sigma_a = 711\,\mathrm{m^{-1}}$ and $\sigma_s = 1861\,\mathrm{m^{-1}}$, respectively

Mathematically, the direct problem in which all the thermophysical and optical properties of the slab are regarded to be known can be described in the following way:

– The governing conservation of energy equation is given by

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\Lambda(T)\frac{\partial T}{\partial x}\right) - \frac{\partial q_r}{\partial x}, \quad 0 < x < E, \quad 0 < t \le t_f, \qquad (1)$$
$$T = T(x,t), \quad q_r = q_r(x,t)$$

where $\partial q_r(x,t)/\partial x$ stands for the source term which represents the divergence of the radiative flux q_r defined by

$$q_r(x,t) = 2\pi \int_{\lambda_{\min}}^{\lambda_{\max}} \int_{\mu=-1}^{\mu=+1} I_\lambda(x,\mu;t)\mu \mathrm{d}\mu \mathrm{d}\lambda \qquad (2)$$

involving the spectral radiation intensities $I_\lambda(x,\mu;t)$ which are determined from the radiative transfer equation (RTE),

$$\underbrace{\frac{1}{c}\frac{\partial I_\lambda}{\partial t}}_{=0} + \mu\frac{\mathrm{d}I_\lambda}{\mathrm{d}x} = \sigma_{a\lambda}(\mu)I_{b,\lambda}(T) - \beta_\lambda(\mu)I_\lambda + \frac{1}{2}\int_{-1}^{+1}\sigma_{s\lambda}(\mu')P(\mu' \to \mu)I_\lambda(x,\mu';t)\mathrm{d}\mu' \qquad (3)$$
$$0 < x < E, \quad 0 < t \le t_f, \quad \mu \in [-1,0) \cup (0,+1],$$
$$\lambda \in [1,100] \times 10^{-6}\,[\mathrm{m}], \quad I_\lambda = I_\lambda(x,\mu;t), \quad T = T(x,t)$$

In Eq. 3, β_λ is the monochromatic extinction coefficient and is a sum of the absorption and scattering coefficients, $I_{b,\lambda}(T)$ is the monochromatic intensity of the blackbody at the absolute temperature T which can be expressed as [8,10]

$$I_{b,\lambda}(T) = \frac{1}{\pi}\frac{2\pi hc^2}{n_\lambda^2\lambda^5\left[\exp\left(\frac{hc}{n_\lambda \lambda k T}\right) - 1\right]}$$
$$= \frac{1.191 \times 10^{-16}}{n_\lambda^2\lambda^5\left[\exp\left(\frac{1.4388\times10^{-2}}{n_\lambda \lambda T}\right) - 1\right]}, [\mathrm{W \cdot m^2 \cdot sr^{-1}}], \lambda\,[\mathrm{m}], T\,[\mathrm{K}] \qquad (4)$$

where n_λ is the monochromatic refractive index of the medium which is equal to unity in this paper, the coefficient c denotes the speed of light $c = 2.998 \times 10^8\ \mathrm{m \cdot s^{-1}}$, and $k = 1.380 \times 10^{-23}\ \mathrm{J \cdot K^{-1}}$ is the Boltzmann constant. The scattering phase function $P(\mu' \to \mu)$ is approximated by the Henyey–Greenstein phase function $P_{\mathrm{HG}}(\cos\theta)$ [8] which can be expanded in a series of Legendre polynomials $P_n(\cos\theta)$ as

$$\begin{aligned} P(\mu' \to \mu) \cong P_{\mathrm{HG}}(\cos\theta) &= \frac{1-g^2}{\left(1+g^2-2g\cos\theta\right)} \\ &= 1 + \sum_{l=1}^{m} (2l+1) g^l P_l(\mu') P_l(\mu) \end{aligned} \tag{5}$$

where $\mu = \cos\theta$ is the direction cosine.

The uniqueness of the problem solution given by Eqs. 1–4 requires a knowledge of initial and boundary conditions. In the case of a combined transient conduction and radiation heat transfer problem, the initial and boundary conditions have to be specified for the temperature field while the boundary conditions are necessary for RTE only. It is assumed in this paper that the face $x = 0$ of the slab is semitransparent for wavelengths of $1\,\mu\mathrm{m} < \lambda < 5\,\mu\mathrm{m}$ and opaque for $5\,\mu\mathrm{m} < \lambda < 100\,\mu\mathrm{m}$ while the face $x = E$ is opaque for all wavelengths in the interval $1\,\mu\mathrm{m} < \lambda < 100\,\mu\mathrm{m}$. In addition to this, there is free convection at the face $x = 0$ with a heat transfer coefficient h_1, and at the face $x = E$, it is treated to be adiabatic. Moreover, for time $0 < t < t_{\mathrm{h}}$, the face $x = 0$ of the slab is heated by the hemispherical surface of the blackbody of constant temperature $T_{\mathrm{b},1} = 400\,\mathrm{K}$. Hence, we have the following initial (IC) and boundary (BC) conditions for the temperature field:

$$\text{IC:} \quad T(x,0) = T_0 \quad \text{for} \quad x \in [0, E] \tag{6}$$

$$\begin{aligned} &\text{BC:} \quad \text{at the face } x = 0 \\ &\Lambda(T)\frac{\partial T}{\partial x}\bigg|_{x=0} = h_1(T - T_{\mathrm{g}1}) + \int_{\lambda=5\,\mu\mathrm{m}}^{\lambda=100\,\mu\mathrm{m}} \varepsilon_1 \pi I_{\mathrm{b},\lambda}(T - T_{\mathrm{g}1})\bigg|_{x=0} \mathrm{d}\lambda \\ &\qquad + (1-\varepsilon_1) \int_{\lambda=5\,\mu\mathrm{m}}^{\lambda=100\,\mu\mathrm{m}} \pi I_{\mathrm{b},\lambda}(T = 400\,\mathrm{K})\mathrm{d}\lambda \end{aligned} \tag{7}$$

$$\text{BC:} \quad \text{at the face } x = E \qquad -\Lambda(T)\frac{\partial T}{\partial x}\bigg|_{x=E} = 0 \tag{8}$$

The boundary conditions for radiation intensities (see [11, p. 548])

BC: at the face $x = 0$

$$I_\lambda(0,\mu,t) = \varepsilon_1 I_{\mathrm{b},\lambda}(T(0,t)) + \tau' \int\limits_{\lambda=1\,\mu\mathrm{m}}^{\lambda=5\,\mu\mathrm{m}} I_{\mathrm{b},\lambda}(T=400\,\mathrm{K})\mathrm{d}\lambda$$
$$+ 2\rho_1^{\mathrm{d}} \int\limits_{-1}^{0} I_\lambda(0,\mu',t)\mu'\mathrm{d}\mu', \quad \mu > 0 \tag{9}$$

where τ' is the transmissivity and ρ_1^{d} stands for the diffuse reflectivity at the face $x = 0$ of the medium.

BC: at the face $x = E$,

$$I_\lambda(E,\mu,t) = \varepsilon_2 I_{\mathrm{b},\lambda}(T(E,t)) + 2(1-\varepsilon_2) \int\limits_{0}^{+1} I_\lambda(E,\mu',t)\mu'\mathrm{d}\mu', \quad \mu < 0 \tag{10}$$

2.2 Inverse problem

In the inverse problem, the thermal conductivity $\Lambda(T)$ and the asymmetry parameter g are unknown, but there are known temperature histories $Y_i(t_n)$ measured at some locations $\{x = x_i : i = 1, 2, \ldots, NMP\}$ of the sample for discrete times $\{t = t_n : n = 1, 2, \ldots, Nt\}$ of a simulated experiment. Assuming that the unknown thermal conductivity $\Lambda(T)$ can be represented in a base of functions $\{1, T, T^2, \ldots, T^K\}$ as

$$\Lambda(T) = \Lambda_0 + \Lambda_1 T + \cdots + \Lambda_K T^K \tag{11}$$

then the parameters to be identified are searched in a form of the vector $\tilde{\mathbf{u}}$ with components

$$\tilde{\mathbf{u}} = [\tilde{\Lambda}_0, \tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_K; \tilde{g}]^T \tag{12}$$

which can be found by minimizing the objective function J as the sum of the squared residuals defined as

$$J(\tilde{\mathbf{u}}^T) = \sum_{i=1}^{NMP} \sum_{n=1}^{Nt} [T_{\mathrm{cal}}(x_i, t_n, \tilde{\mathbf{u}}^T) - Y_i(t_n)]^2 \tag{13}$$

where $T_{\mathrm{cal}}(x_i, t_n; \tilde{\mathbf{u}}^T)$ and $Y_i(t_n)$ are the temperatures calculated from DP (Eqs. 1–10) upon the fixed vector $\tilde{\mathbf{u}}$ (for a current iteration number (s)) and the measured values from a simulated experiment, respectively [4,5].

Due to the strong nonlinearity of the boundary-value problem given by Eqs. 1–10, the minimum of the objective function J—Eq. 13, can be found iteratively. Assuming the initial values of the components of the estimated vector $\tilde{\mathbf{u}}$ as $\tilde{\mathbf{u}}^{(0)}$, then the iterative procedure of minimization of the objective function J can be written conveniently in matrix form as [1]

$$\tilde{\mathbf{u}}^{(s+1)} = \tilde{\mathbf{u}}^{(s)} + \left[\mathbf{X}^T(\tilde{\mathbf{u}}^{(s)})\mathbf{X}(\tilde{\mathbf{u}}^{(s)})\right]^{-1} \mathbf{X}^T(\tilde{\mathbf{u}}^{(s)})[\mathbf{T}_{\text{cal}}(\tilde{\mathbf{u}}^{(s)}) - \mathbf{Y}] \tag{14}$$

where s is the successive number of the iteration, and $\mathbf{X}^T(\tilde{\mathbf{u}}^{(s)})$ denotes the transposition of the matrix of sensitivity coefficients with elements X_{inj} defined as

$$X_{inj} = \frac{\partial T_{\text{cal}}(x_i, t_n; \tilde{\mathbf{u}}^T)}{\partial \tilde{u}_j}, \quad i = 1, 2, \ldots, NMP; \quad n = 1, 2, \ldots, Nt; \quad j = 0, 1, \ldots, K+1 \tag{15}$$

and $[\mathbf{T}_{\text{cal}}(\tilde{\mathbf{u}}^{(s)}) - \mathbf{Y}]$ is the residual vector with components arranged as follows:

$$[\mathbf{T}_{\text{cal}} - \mathbf{Y}] \equiv [T(x_1, t_1) - Y_1(t_1), \ldots, T(x_1, t_{Nt}) - Y_1(t_{Nt}), \ldots, T(x_{NMP}, t_1) - Y_{NMP}(t_1), \ldots, T(x_{NMP}, t_{Nt}) - Y_{NMP}(t_{Nt})]^{\text{T}} \tag{16}$$

The sensitivity coefficients (Eq. 15) can be calculated either by solving the sensitivity problem with respect to DP [1] or by using a finite difference scheme which has been carried out in this paper. The central difference scheme applied here can be written as

$$X_{inl} = \frac{T_{\text{cal}}(y_i, t_n; \tilde{u}_1, \ldots, (\tilde{u}_l + \delta\tilde{u}_l), \ldots, \tilde{u}_{K+1}) - T_{\text{cal}}(y_i, t_n; \tilde{u}_1, \ldots, (\tilde{u}_l - \delta\tilde{u}_l), \ldots, \tilde{u}_{K+1})}{2\delta\tilde{u}_l} + O(\delta\tilde{u}_l^2) \tag{17}$$

where $\delta\tilde{u}_l = EPS \cdot \tilde{u}_l$, $EPS \in [10^{-5}, 10^{-3}]$. It has been recommended by Minkowycz et al. [2, p. 450] to calculate the sensitivity coefficients as

$$X_{inl} = \frac{\text{Im}[T_{\text{cal}}(y_i, t_n; \tilde{u}_1, \ldots, (\tilde{u}_l + i \cdot \delta\tilde{u}_l), \ldots, \tilde{u}_{K+1})]}{\delta\tilde{u}_l} + \text{O}\,(\delta\tilde{u}_l^2) \tag{18}$$

where Im[$\cdot$] stands for the imaginary part of the estimated temperature $T_{\text{cal}}(\cdot)$ and $i = \sqrt{(-1)}$ is the imaginary unit. The better performance of the scheme of Eq. 18 over the scheme of Eq. 17 gives, in fact, that the relative error in Eq. 18 is independent of applying a step size $\delta\tilde{u}_l$.

The iterative procedure of minimization of the objective function J (Eq. 14) is realized by the Newton–Gauss method so its convergence is dependent on an initial choice of the starting point $\tilde{\mathbf{u}}^{(0)}$. If, for example, the estimated values $\tilde{\mathbf{u}}^{(0)}$ has been chosen too far away from the exact solution, then in the subsequent iteration, the new solution moves away from the exact solution and the iterative procedure (Eq. 14) becomes divergent. On the other hand, if there are columns of the matrix of sensitivity coefficient $\mathbf{X}$ which are nearly linearly dependent on each other, then the matrix $[\mathbf{X}^T\mathbf{X}]$ becomes singular (determinant $\det[\mathbf{X}^T\mathbf{X}] = 0$ for a given computer precision) and it is impossible in such a situation to calculate the inverse $[\mathbf{X}^T\mathbf{X}]^{-1}$ which leads finally to the divergence of the iterative procedure (Eq. 14). To diminish the influence of these effects on the procedure convergence and stability, the Levenberg–Marquardt method [1] is recommended and was used in this paper. The modified procedure (Eq. 14) can be written in the following form:

$$\tilde{\mathbf{u}}^{(s+1)} = \tilde{\mathbf{u}}^{(s)} + \left[\mathbf{X}^T(\tilde{\mathbf{u}}^{(s)})\mathbf{X}(\tilde{\mathbf{u}}^{(s)}) + \gamma^{(s)}\mathbf{I}\right]^{-1} \mathbf{X}^T(\tilde{\mathbf{u}}^{(s)})[\mathbf{T}_{\text{cal}}(\tilde{\mathbf{u}}^{(s)}) - \mathbf{Y}] \tag{19}$$

which differs from Eq. 14 by adding a stabilization term $\gamma^{(s)}\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and $\gamma^{(s)}$ denotes the damping parameter that is getting smaller if the value of the objective function at iteration $(s + 1)$ is less than at iteration (s), and vice versa.

An accepted measure of the linear dependence of the columns of the matrix $[\mathbf{X}^T\mathbf{X}+\gamma\mathbf{I}]$ are the correlation coefficients $r_{i,j}$ which can be expressed as

$$r_{i,j} = \frac{\operatorname{cov}(u_i, u_j)}{\sigma_i\sigma_j} \approx \frac{P_{i,j}}{\sqrt{P_{i,i}}\sqrt{P_{j,j}}} \tag{20}$$

where σ_i, σ_j denote the standard deviations of the measurement errors and $P_{i,j}$'s are the elements of the matrix

$$P_{i,j} = [\mathbf{X}^T(\tilde{\mathbf{u}})\mathbf{X}(\tilde{\mathbf{u}}) + \gamma\mathbf{I}]^{-1}_{i,j} \tag{21}$$

A knowledge of the values of the correlation coefficients obtained for various times t_{f} of an experiment duration is essential for a stage of experiment planning, but it is not a subject of this paper.

To estimate the quality of the iterative procedure (Eq. 19) used for simultaneous identification of the thermal conductivity $\Lambda(T)$ and the asymmetry parameter g, the following measures were accepted:

– measure of difference between the exact $\hat{\Lambda}(T)$ and estimated $\tilde{\Lambda}(T)$ values of the thermal conductivity

$$d(\Lambda, T_{\min}, T_{\max}) = \int_{T_{\min}}^{T_{\max}} \left|(\tilde{\Lambda}(T) - \hat{\Lambda}(T)\right| \mathrm{d}T \tag{22}$$

– confidence intervals for the estimated parameters at a 99 % confidence level (assuming that the measurement errors are normally distributed random variables with zero mean and unit standard deviation)

$$\hat{u}_i - 2.576\sigma_i \le \tilde{u}_i \le \hat{u}_i + 2.576\sigma_i, \quad i = 0, 1, \ldots, K+1 \tag{23}$$

3 Numerical Treatment

The thermal conductivity to be identified $\Lambda(T)$ has been derived from the literature [10,12]. Its thermal characteristics is given in the form,

$$\hat{\Lambda}(T) = 2.572 \times 10^{-4} T^{0.81} + 5.27 \times 10^{-5} \rho^{0.91} (1 + 0.0013T) \tag{24}$$

which describes the heat transfer due to conduction in fibrous insulation made of silica fibers and air. On the other hand, the exact value of the asymmetry parameter has been accepted as $\hat{g} = 0.4$.

3.1 Solution of the Energy Equation

To solve the energy equation, the finite volume method (FVM) was used. Based on the discretization grid ω_h

$$\omega_\mathrm{h} = \Big\{ (x_i, t_n) : x_i = i\Delta x, t_n = n\Delta t, \\ \Delta x = E/(Nx+1), \Delta t = t_\mathrm{f}/Nt, i = \overline{0, Nx+1}, n = \overline{0, Nt} \Big\} \tag{25}$$

the energy equation can be expressed as

$$\rho c_p \frac{T_i^{n+1} - T_i^n}{\Delta t} = \Lambda_{i+1/2} \frac{T_{i+1}^{n+1} - T_i^{n+1}}{(\Delta x)^2} - \Lambda_{i-1/2} \frac{T_i^{n+1} - T_{i-1}^{n+1}}{(\Delta x)^2} + (S_\mathrm{r})_i^{n+1}, \\ i = \overline{1, Nx}, \ n = \overline{0, Nt-1} \tag{26}$$

where

$$T_i^n = T(x_i, t_n), \quad \Lambda_{i+1/2} = 0.5(\Lambda(T_i^{n+1}) + \Lambda(T_{i+1}^{n+1})), \quad \Lambda_{i-1/2} = 0.5(\Lambda(T_i^{n+1}) \\ + \Lambda(T_{i-1}^{n+1})), \quad (S_\mathrm{r})_i^{n+1} = -(\mathrm{d}q_\mathrm{r}/\mathrm{d}x)_i^{n+1} \tag{27}$$

and is subject to the following initial conditions,

$$T_i^0 = T_0, \quad i = \overline{0, Nx+1} \tag{28}$$

and boundary conditions,

$$\Lambda_{1/2} \frac{T_1^{n+1} - T_0^{n+1}}{\Delta x} = (\rho c_p) \frac{T_0^{n+1} - T_0^n}{\Delta t} \frac{\Delta x}{2} + h_1 (T_0^{n+1} - T_{g1}) \\ + \int_{\lambda=5\,\mu\mathrm{m}}^{\lambda=100\,\mu\mathrm{m}} \varepsilon_1 \pi I_{\mathrm{b},\lambda} (T_0^{n+1} - T_{g1}) \mathrm{d}\lambda \\ + (1-\varepsilon_1) \int_{\lambda=5\mu\mathrm{m}}^{\lambda=100\mu\mathrm{m}} \pi I_{\mathrm{b},\lambda} \, (T = 400\,\mathrm{K}) \mathrm{d}\lambda, \quad n = \overline{0, Nt-1} \tag{29}$$

$$-\Lambda_{Nx+1/2} \frac{T_{Nx+1}^{n+1} - T_{Nx}^{n+1}}{\Delta x} = (\rho c_p) \frac{T_{Nx+1}^{n+1} - T_{Nx+1}^n}{\Delta t} \frac{\Delta x}{2}, \quad n = \overline{0, Nt-1} \tag{30}$$

3.2 Solution of RTE

A finite difference method (FDM) and DOM applied by Asllanaj et al. [12] to solve RTE, which is a coupled system of nonlinear integro-differential equations with respect to the spectral intensity $I_\lambda(x, \mu, t)$ and temperature $T(x, t)$, was used here to find the source term S_r. One can find the details in [12–14], and there is no need to repeat them here. However, the boundary conditions for the intensity accepted in this paper differ from those given in [12], so the current system of equations has the form,

$$\mathbf{G}_\lambda \cdot \mathbf{I}_\lambda = \mathbf{F}_\lambda \tag{31}$$

where $\mathbf{G}_\lambda$ is a square matrix of dimension $(Nx + 2) \times (Nx + 2)$ with elements,

$$\mathbf{G}_\lambda = \begin{bmatrix} \mathbf{G}_\lambda^6 & & & & & & & \\ -\mathbf{G}_\lambda^6 & \mathbf{G}_\lambda^1 & \mathbf{G}_\lambda^4 & \mathbf{G}_\lambda^5 & & & & \\ -\mathbf{I} & \mathbf{G}_\lambda^2 & \mathbf{I} & & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & & & \\ & & & & -\mathbf{I} & \mathbf{G}_\lambda^2 & \mathbf{I} & \\ \mathbf{0} & & & & \mathbf{G}_\lambda^6 & \mathbf{G}_\lambda^7 & \mathbf{G}_\lambda^3 & -\mathbf{G}_\lambda^5 \\ & & & & & & & -\mathbf{G}_\lambda^5 \end{bmatrix} \tag{32}$$

Here, all the block matrixes $\mathbf{G}_\lambda^1, \mathbf{G}_\lambda^2, \ldots, \mathbf{G}_\lambda^7$ are the same as in [12] but only the first and last elements of the vector $\mathbf{F}_\lambda$ are different. They are given by

$$\begin{aligned} &\mathbf{F}_\lambda^1(0, \mu, t_{n+1}) \\ &\quad = \left(\varepsilon_1 I_{\mathrm{b},\lambda}(T_0^{n+1}) + \tau' I_{\mathrm{b},\lambda}(T = 400\,\mathrm{K}) + 2\rho_1^{\mathrm{d}} \sum_{k=m/2+1}^{m} I_\lambda(0, \mu_k, t_{n+1})\, \mu_k w_k \right) \\ &\qquad \cdot \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{33}$$

$$\begin{aligned} &\mathbf{F}_\lambda^{Nx+2}(E, \mu, t_{n+1}) \\ &\quad = \left(\varepsilon_2 I_{\mathrm{b},\lambda}(T_{Nx+1}^{n+1}) + 2(1 - \varepsilon_2) \sum_{k=1}^{m/2} I_\lambda(E, \mu_k, t_{n+1})\, \mu_k w_k \right) \cdot \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \end{aligned} \tag{34}$$

where μ_k, w_k are the nodes and weights of Gauss-Legendre quadrature, respectively.

4 Results of Numerical Simulations

For performing calculations, the following parameters listed in Table 1 were used.

Table 1 Input data for numerical simulations

Thickness of the planar slab: $E = 0.040\,\mathrm{m}$
Density of the fibrous medium: $\rho = 20\,\mathrm{kg \cdot m^{-3}}$ and its specific heat: $c_p = 670\,\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$
Mean absorption coefficient: $\sigma_a = 711\,\mathrm{m^{-1}}$, mean scattering coefficient: $\sigma_s = 1861\,\mathrm{m^{-1}}$
Refractive index: $n = 1.0$
Transmissivity within the spectral range [1 μm, 5 μm]: $\tau' = 0.05$
Heat transfer coefficients: $h_1 = 5\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$, $h_2 = 0\,\mathrm{W \cdot m^{-2} \cdot K^{-1}}$
Initial temperature: $T_0 = 300\,\mathrm{K}$, and surroundings temperature $T_{g1} = 300\,\mathrm{K}$
Spectral range $\lambda \in [1, 5] \cup (5, 100]$ μm
Number of spatial intervals: $(Nx + 1) = 20$ with equal lengths $\Delta x = 0.002\,\mathrm{m}$
Number of time intervals: $Nt = 100$ or $Nt = 200$ with equal lengths $\Delta t = 0.2\,\mathrm{s}$
Number of discrete angular directions: $m = 8$
Time of heating: $t_h = 3.0\,\mathrm{s}$ and final time: $t_f = 20.0\,\mathrm{s}$ or $t_f = 40.0\,\mathrm{s}$
Number and location of the measuring points: $NMP = 4$, $y_i \in \{2.0, 10.0, 20.0, 32.0\}$ mm
Number of unknown parameters: $(K + 1) = 4$ or $(K + 1) = 3$
Initial estimated values of unknown parameters: $\tilde{\mathbf{u}}^{(0)} = [10^{-2}, 10^{-4}, 10^{-6}; 0.3]$ or $\tilde{\mathbf{u}}^{(0)} = [10^{-2}, 10^{-4}; 0.3]$
Magnitude of disturbance of measuring temperatures: $ZAB \in \{0.01, 0.05, 0.10\}$

The simulated temperature responses $Y_i(t_n)$ at the given locations $\{y_1, y_2, y_3, y_4\}$ are obtained by adding normally distributed errors $N(0, 1)$ to the exact solutions of the boundary-value problem given by Eqs. 1–10 as

$$Y_i(t_n) = \hat{Y}_i(t_n) + ZAB \cdot N(0, 1), \quad i = 1, 2, \ldots, NMP; \; n = 1, 2, \ldots, Nt \tag{35}$$

where ZAB is the standard deviation of measured temperatures.

The transient solution of DP (Eqs. 1–10) for the exact data at the given locations y_i is shown in Fig. 2, and the temperature distributions within the medium for the times $t = 8\,\mathrm{s}$, $t = 20\,\mathrm{s}$, and $t = 40\,\mathrm{s}$ are shown in Fig. 3. The reduced sensitivity coefficients, defined as

$$\bar{X}_{inj} = \tilde{u}_i \frac{\partial T_{\mathrm{cal}}(x_i, t_n; \tilde{\mathbf{u}}^T)}{\partial \tilde{u}_i}, \quad i = 1, 2, \ldots, NMP; \; n = 1, 2, \ldots, Nt; \; j = 0, 1, \ldots, (K + 1) \tag{36}$$

are presented in Fig. 4 for the case of linear dependence of the thermal conductivity $\Lambda(T)$ on temperature.

One can observe in Fig. 4 that the largest value of the reduced sensitivity coefficient at the location y_1 is achievable at the end of heating ($t = 3\,\mathrm{s}$) for the first of the estimated parameters Λ_0 and then for the second one Λ_1. The correlation coefficients r_{ij}, calculated from Eq. 20, where, e.g., r_{12} denotes the measure of strength of the linear dependence between the first $\tilde{u}_0 = \Lambda_0$ and the second of the estimated parameters

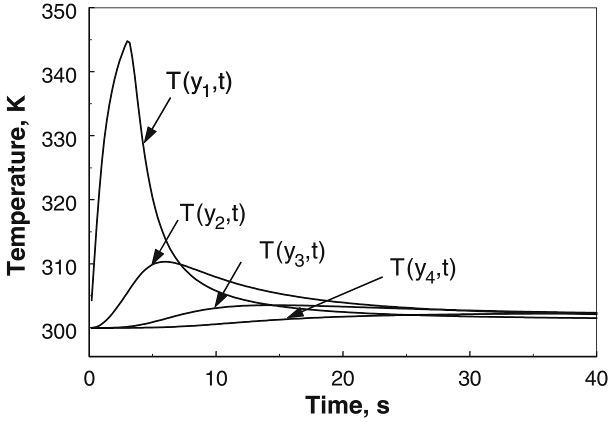

Fig. 2 Transient solution of DP at the given locations of measuring points y_i

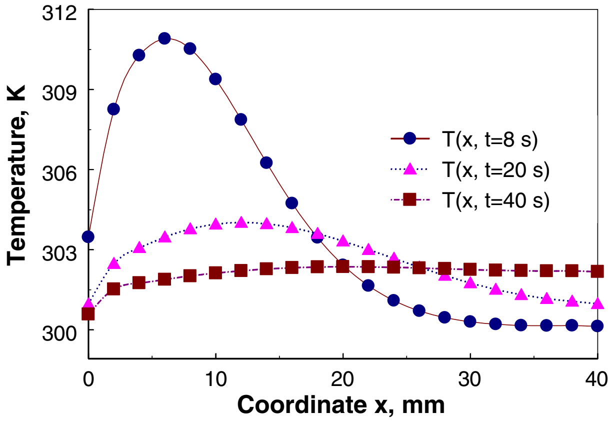

Fig. 3 Temperature distribution within the medium

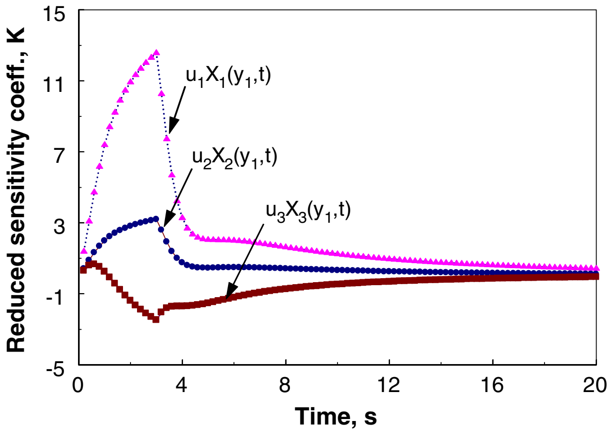

Fig. 4 Reduced sensitivity coefficients at location $y_1 = 2\,\text{mm}$

Table 2 Results of parameter estimation (in brackets just after the sign ± are given the width of the confidence intervals of the estimated parameters, the exact asymmetry parameter $g = 0.4$)

$ZAB = 0.01$:	$\Lambda_0 = (0.5260 \times 10^{-2} \pm 0.2276 \times 10^{-2})$, $\Lambda_1 = (0.7119 \times 10^{-4} \pm 0.6179 \times 10^{-5})$, $g = (0.4030 \pm 0.6152 \times 10^{-2})$
$ZAB = 0.05$:	$\Lambda_0 = (0.9722 \times 10^{-2} \pm 0.1245 \times 10^{-1})$, $\Lambda_1 = (0.5894 \times 10^{-4} \pm 0.3381 \times 10^{-4})$, $g = (0.3888 \pm 0.3409 \times 10^{-1})$
$ZAB = 0.10$:	$\Lambda_0 = (-0.1293 \times 10^{-1} \pm 0.5537 \times 10^{-1})$, $\Lambda_1 = (0.1203 \times 10^{-3} \pm 0.1500 \times 10^{-3})$, $g = (0.4514 \pm 0.1376)$
$ZAB = 0.10 +$ appr:	$\Lambda_0 = (0.6088 \times 10^{-2} \pm 0.6318 \times 10^{-2})$, $\Lambda_1 = (0.6882 \times 10^{-4} \pm 0.1715 \times 10^{-4})$, $g = (0.4038 \pm 0.1719 \times 10^{-1})$
$d(\Lambda, 300, 345; ZAB = 0.01) = 0.2433$,	$d(\Lambda, 300, 345; ZAB = 0.05) = 7.1057$,
$d(\Lambda, 300, 345; ZAB = 0.10) = 181.2888$,	$d(\Lambda, 300, 345; ZAB = 0.10 + \text{appr.}) = 1.8181$

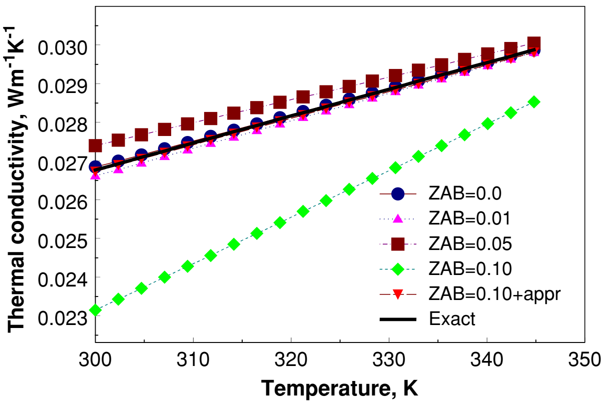

Fig. 5 Results of estimation of the thermal conductivity $\Lambda(T)$

$\tilde{u}_1 = \Lambda_1$, etc., are equal to $r_{12} = -0.9999$, $r_{13} = -0.9581$, and $r_{23} = +0.9584$, respectively. One can draw a conclusion that all of the estimated parameters are highly correlated to each other. In this case, the linear dependence of the thermal conductivity $\Lambda(T)$ on temperature, i.e., $\Lambda(T) = \Lambda_0 + \Lambda_1 T$, is the highest order of polynomial representation of the thermal-conductivity dependence on temperature which can be taken into account using the estimating procedure (Eq. 19). In Table 2, the results of numerical estimation are given for various values of the parameter *ZAB*.

In Table 2, the abbreviation *"appr."* means that the simulated temperature responses $Y_i(t_n)$ were smoothed numerically using the double precision procedures DCSSCV and DCSVAL derived from the IMSL numerical library [15]. The results of parameter estimation are presented graphically in Fig. 5.

5 Conclusions

The inverse problem of simultaneous identification of the temperature-dependent thermal conductivity and the asymmetry parameter g of the Henyey–Greenstein scattering phase function has been solved numerically for the case of transient one-dimensional heat transfer in a plane-parallel grey participating medium. The problem has been formulated for a more general case in which the monochromatic optical properties of the medium can be used. The obtained results of calculation allow one to draw the following conclusions:

– Simultaneous identification of the thermal conductivity $\Lambda(T)$ and the asymmetry parameter g of the Henyey–Greenstein phase function for a transient conductive–radiative heat transfer problem is possible by using an inverse method.
– Identification of the thermal conductivity temperature dependence $\Lambda(T)$ is possible by assuming only a linear form $\Lambda(T) = \Lambda_0 + \Lambda_1 T$ due to high correlation between the coefficients Λ_0 and Λ_1 (Table 2).
– Initial smoothing of measuring temperatures by cubic splines using the cross-validation method to estimate the smoothing parameter (IMSL procedures DCSSCV and DCSVAL) is recommended if the measurement errors are greater than 0.1 K.

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